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Numerical Computation of Wave Localization in Large Disordered Beam-Like Lattice Trusses

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Wave localization in randomly disordered multiwave structures is investigated. The smallest localization factor is of particular interest and is related to the smallest positive Lyapunov exponent. The numerical algorithm by Wolf et al. is modified to determine all of the Lyapunov exponents. The wave localization in a large beamlike lattice truss modeled as an equivalent continuous Timoshenko beam is studied.

I. Introduction

MANY engineering structures are designed to be composed of identically constructed elements assembled end to end to form a spatially periodic structure, such as long space antennae or large beamlike lattice trusses used in space solar power stations. Periodic structures behave like bandpass filters when propagating stress waves. If damping is neglected, waves are classified as traveling waves that propagate without attenuation and nontraveling or attenuating waves whose amplitudes attenuate as the waves propagate. For single-wave periodic structures, the frequency axis is divided into alternating passbands and stopbands. In the frequency passbands, the wave is a traveling wave, whereas in the frequency stopbands, the wave is an attenuating wave. For multiwave structures, in a given frequency band, some of the waves may be traveling waves and the others nontraveling waves.

However, due to defects in manufacture and assembly, no structure designed as a periodic structure can be perfectly periodic. Disorder can occur in the geometry of configurations and material properties of the structure.

In a disordered periodic structure, wave amplitudes of all waves will be attenuated, even those that are traveling waves in the perfectly periodic counterpart. This means that the vibrational energy imparted to the structure by an external source cannot propagate to arbitrarily long distances but is instead substantially confined to a region close to the source. This phenomenon is known as wave localization. In disordered periodic structures, it is therefore of great practical importance to study the localization behavior and evaluate the so-called localization factors, namely, the exponential rates at which the amplitudes of the waves propagating in the structures decay. The reciprocal of the localization factor gives the localization length, which characterizes the distance that the propagating wave extends in the structure. As an application, the range of damage that is spread in a structure due to impact at some location can be estimated.

There are d waves in a d-wave disordered periodic structure; each wave attenuates at a certain exponential rate or corresponds to a certain localization factor, which implies that each wave will extend to a certain localization length. The smallest localization factor or the largest localization length is of particular interest for multiwave structures, since it corresponds to the wave that has potentially the least amount of decay or that carries energy along the multiwave structure farther than any other waves.

The study of the localization phenomenon has been and remains an active research area in solid state physics after the celebrated work of Anderson.[1] Hodges[2] was the first to recognize the relevance of localization theory to dynamical behavior of periodic engineering structures. Since then, there have been several studies on the localization of vibration of structures. In a series of publications, Pierre and Dowell,[3] Pierre,[4] and Cha and Pierre[5] studied the localization phenomenon for monocoupled disordered structural systems. Cai and Lin[6] developed a perturbation scheme to calculate the localization factor based on a generic periodic structure.

Kissel[7] and Ariaratnam and Xie[8,9] used a traveling wave approach to investigate the localization effects in one-dimensional periodic engineering structures. A transfer matrix formulation, including wave transfer matrices, was used to model disordered periodic structures. Furstenberg's theorem[10] for products of random matrices was applied to calculate the localization effect as a function of frequency. The localization factor was related to the largest Lyapunov exponent. In these publications, only one-dimensional or monocoupled periodic structures were considered, in which the transfer matrices are of dimension 2.

Few studies have been done on the localization behavior of multiwave periodic structures. Kissel[11] derived the localization factor as a function of the transmission matrix for multiwave disordered systems using the multiplicative ergodic theorem of Oseledec.[12] The difficulty in the study of multiwave periodic structures is that the localization factor is related to the smallest positive Lyapunov exponent and not the largest Lyapunov exponent as in the single-wave case. The numerical algorithms used previously for evaluating the largest Lyapunov exponent for single-wave periodic structures cannot be employed for multiwave structures.

In this paper, a modification of an algorithm due to Wolf et al.[13] is used to determine all of the Lyapunov exponents for randomly disordered multiwave periodic structures. The localization factor of a large beamlike lattice truss, modeled as an equivalent continuous Timoshenko beam, is evaluated.

II. Transfer Matrices and Lyapunov Exponents for Multiwave Structures

Consider an element numbered n in a multiwave structure (Fig. 1). The element is modeled by the transfer matrix $\boldsymbol{T}_n$, which relates a state vector $\boldsymbol{x}_{n-1}$ on the left-side of element n to that on the right-side, $\boldsymbol{x}_n$, by the linear transformation

$$\boldsymbol{x}_n = \boldsymbol{T}_n \boldsymbol{x}_{n-1} \tag{1}$$

In linearly elastic engineering structures, the state vector $\boldsymbol{x}_n$ usually consists of generalized displacements and generalized forces, i.e., $\boldsymbol{x}_n = \{u_{n1}, u_{n2}, \ldots, u_{nd}; f_{n1}, f_{n2}, \ldots, f_{nd}\}^T$ for a d-wave structure, where $u_{n1}, u_{n2}, \ldots, u_{nd}$ are generalized displacements, and $f_{n1}, f_{n2}, \ldots, f_{nd}$ are generalized forces. The transfer matrix can be derived from the dy-

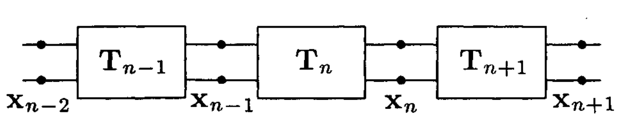

Fig. 1 Transfer matrices.

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namic equations of motion of the structure by, possibly, the finite difference or finite element method. The derivation of transfer matrices of various engineering structures is presented in detail by Pestel and Leckie.[14] Since the dimension of the state vector is $2d$, the transfer matrix is a square matrix of dimension $2d \times 2d$ for a d-wave structure. If the structure is symmetric, the transition from the section n to the section $n+1$ is the same as that from $n+1$ to n. Hence it follows that $\det(T_n) = \det(T_n^{-1})$, so that $\det(T_n) = 1$, where $\det(T_n)$ denotes the determinant of T_n.

For a perfectly periodic structure, each element of the structure is identical; therefore, the transfer matrix for each element is the same, i.e., $T_n = T$ for all n. The state vector after n elements is related to that at the beginning by

$$x_n = T^n x_0 \tag{2}$$

When the periodic structure is disordered randomly due to variability in geometry, material, and manufacturing conditions, the transfer matrix for each element is not the same but is a function of the parameters of disorder. In this case, the state vector after n elements is related to the state vector at the beginning by

$$x_n = T_n T_{n-1} \dots T_1 x_0 \tag{3}$$

where $T_1, T_2, \dots, T_n$ are random matrices.

A Lyapunov exponent is defined by

$$\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \ln \|x_n(x_0)\| \tag{4}$$

where $\|\cdot\|$ is the Euclidean vector norm. The Lyapunov exponent describes the average rate of exponential decay or growth of state vector x_0. In general, there will be several Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2d}$, where $2d$ is the dimension of the matrices $T_1, T_2, \dots$.

To obtain the properties of the Lyapunov exponents, it is easier to use the wave transfer matrix formulation. Much of the discussion of multiwave in this section follows Ref. 11. The wave transfer matrix W_n is a linear transformation relating the left (or backward) and the right (or forward) traveling wave amplitudes, A^L and B^R, at two neighboring sections of the element n of the structure (Fig. 2),

$$\begin{Bmatrix} A_n^L \\ B_n^R \end{Bmatrix} = W_n \begin{Bmatrix} A_{n-1}^L \\ B_{n-1}^R \end{Bmatrix} \tag{5}$$

where $A_n^L = \{A_{n1}^L, A_{n2}^L, \dots, A_{nd}^L\}^T$, and $B_n^R = \{B_{n1}^R, B_{n2}^R, \dots, B_{nd}^R\}^T$.

The wave transfer matrix can be derived from the transfer matrix. Let the eigenvalues of the transfer matrix of the perfectly periodic structure be $e^{\pm ik_1}, e^{\pm ik_2}, \dots, e^{\pm ik_d}$. If k_j is real, the corresponding jth wave is a traveling or propagating wave, the positive sign indicating a left-traveling wave and the negative sign a right-traveling wave. The real number k_j is the wave number. If k_j is imaginary, the jth wave is a nontraveling or attenuating wave. For a perfectly periodic system, the wave transfer matrix is of the form

$$W_n = \begin{bmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{bmatrix} \tag{6}$$

where $\Lambda_+ = \mathrm{diag}\,\{e^{\pm ik_1}, e^{\pm ik_2}, \dots, e^{\pm ik_d}\}$, and $\Lambda_- = \mathrm{diag}\,\{e^{-ik_1}, e^{-ik_2}, \dots, e^{-ik_d}\}$.

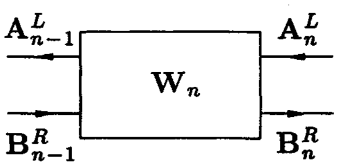


Fig. 2 Wave transfer matrix.

As defined in Eq. (1), the transfer matrix T_n relates a state vector x_{n-1} on the left side of element n to that on the right side, x_n, i.e.,

$$\begin{Bmatrix} u_n \\ f_n \end{Bmatrix} = T_n \begin{Bmatrix} u_{n-1} \\ f_{n-1} \end{Bmatrix}, \qquad T_n = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{7}$$

On the other hand, the generalized displacements of the perfectly periodic system may be expressed in terms of the wave amplitudes

$$\begin{Bmatrix} u_n \\ u_{n-1} \end{Bmatrix} = Q \begin{Bmatrix} A_n^L \\ B_n^R \end{Bmatrix} \tag{8}$$

where

$$Q = \begin{bmatrix} I & I \\ \Lambda_- & \Lambda_+ \end{bmatrix}$$

From Eq. (7), it can be shown that

$$\begin{Bmatrix} u_n \\ f_n \end{Bmatrix} = U \begin{Bmatrix} u_n \\ u_{n-1} \end{Bmatrix} \tag{9}$$

where

$$U = \begin{bmatrix} I & 0 \\ T_{22}T_{12}^{-1} & T_{21} - T_{22}T_{12}^{-1}T_{11} \end{bmatrix}$$

where I is the $d \times d$ unit matrix.

Hence, from Eqs. (8) and (9), the state vector x_n is related to left- and right-traveling wave amplitudes by

$$\begin{Bmatrix} u_n \\ f_n \end{Bmatrix} = X \begin{Bmatrix} A_n^L \\ B_n^R \end{Bmatrix} \tag{10}$$

where $X = UQ$.

Equations (9) and (10) imply that

$$W_n = X^{-1} T_n X \tag{11}$$

From Eq. (6), it may be seen that X is the matrix of eigenvectors of the transfer matrix of the perfectly periodic structure. It is noted that Eq. (11) is valid for both perfectly periodic and disordered structures. When the transfer matrices are random, the eigenvector matrix X will be that for the average transfer matrix.

When a periodic structure is disordered, the wave amplitudes after n elements are related to those at the beginning through a matrix in the form of a product of random matrices:

$$\begin{Bmatrix} A_n^L \\ B_n^R \end{Bmatrix} = V_n \begin{Bmatrix} A_0^L \\ B_0^R \end{Bmatrix} = W_n W_{n-1} \dots W_1 \begin{Bmatrix} A_0^L \\ B_0^R \end{Bmatrix} \tag{12}$$

where $W_1, W_2, \dots, W_n$ are the wave transfer matrices of the individual elements.

It can be shown that the Lyapunov exponents are given by[15]

$$\lambda_i = \lim_{n \to \infty} \frac{1}{n} \ln \sigma_i(V_n), \qquad i = 1, 2, \dots, 2d \tag{13}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{2d}$ are the singular values of the matrix V_n given by Eq. (12), i.e., $\sigma_i(V_n)$ is equal to the positive square root of the ith eigenvalue of the matrix $V_n^* V_n$, * denoting transposition and taking complex conjugate.

The relationships among left- and right-traveling waves can also be expressed in terms of the scattering matrix S_n of element n:

$$\begin{Bmatrix} A_{n-1}^L \\ B_n^R \end{Bmatrix} = S_n \begin{Bmatrix} B_{n-1}^L \\ A_n^R \end{Bmatrix}, \qquad S_n = \begin{bmatrix} r_n & t_n \\ \hat{t}_n & \hat{r}_n \end{bmatrix} \tag{14}$$

where t_n, $\hat{t}_n$, r_n, and $\hat{r}_n$ are the transmission and reflection matrices, respectively. The corresponding wave transfer matrix W_n can be obtained by rearranging the state vectors of Eq. (14):

$$\begin{Bmatrix} A_n^L \\ B_n^R \end{Bmatrix} = W_n \begin{Bmatrix} A_{n-1}^L \\ B_{n-1}^R \end{Bmatrix} = \begin{bmatrix} t_n^{-1} & -t_n^{-1} r_n \\ \hat{r}_n t_n^{-1} & \hat{t}_n - \hat{r}_n t_n^{-1} r_n \end{bmatrix} \begin{Bmatrix} A_{n-1}^L \\ B_{n-1}^R \end{Bmatrix} \tag{15}$$

It is usually assumed that the scattering matrix S_n of a disordered element is symmetric. The symmetry of the scattering matrix follows from the symmetry of the impedance (or admittance matrix) of the element. The symmetry of S_n implies that

$$r_n^T = r_n, \qquad \hat{r}_n^T = \hat{r}_n, \qquad t_n^T = \hat{t}_n \tag{16}$$

The conditions (16) lead to the symplecticity of the wave transfer matrix W_n, i.e.,

$$W_n^T J W_n = J$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Since the wave transfer matrices $W_1, W_2, \ldots, W_n$ are symplectic matrices, $V_n = W_n W_{n-1} \ldots W_1$ is also symplectic. It can be easily shown that the singular values of a symplectic matrix occur in reciprocal pairs, i.e.,

$$\sigma_1, \sigma_2, \ldots, \sigma_d, \sigma_d^{-1}, \ldots, \sigma_2^{-1}, \sigma_1^{-1}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_d \geq 1$. Therefore, the Lyapunov exponents given by Eq. (13) have the following property:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq (\lambda_{d+1} = -\lambda_d)$$
$$\geq (\lambda_{d+2} = -\lambda_{d-1}) \geq \ldots \geq (\lambda_{2d} = -\lambda_1) \tag{17}$$

The d positive Lyapunov exponents correspond to the right-traveling waves, whereas the d negative Lyapunov exponents correspond to the left-traveling waves. It can be seen that the localization factors are related to the Lyapunov exponents by definition, since both quantities characterize the average exponential decaying rates of the amplitudes of propagating waves. The smallest positive Lyapunov exponent λ_d is of particular interest for multiwave structures, since λ_d represents the wave that has potentially the least amount of decay or that carries energy along the multiwave structure farther than any other waves.

From the well-known multiplicative ergodic theorem by Oseledec,[12] it is known that a randomly chosen initial state vector x_0 will evolve in general at the rate of the largest Lyapunov exponent λ_1. It is therefore relatively easy to determine the largest Lyapunov exponent λ_1. Since the smallest positive Lyapunov exponent λ_d is of interest for a d-wave structure, the previous evolution property gives rise to the difficulty in the study of the localization behavior of multiwave structures. In the next section, an algorithm will be introduced to evaluate all of the Lyapunov exponents λ_i, $i = 1, 2, \ldots, 2d$, of a d-wave periodic structure.

III. Algorithm for Calculating All Lyapunov Exponents

The calculation of all of the Lyapunov exponents for systems whose equations of motion are explicitly known was investigated by Benettin et al.[16] and Wolf et al.[13] These algorithms are designed for determining all of the Lyapunov exponents for continuous dynamical systems, which are not directly suitable for discrete dynamical systems. In this section, the algorithm due to Wolf et al.[13] is modified to determine all of the Lyapunov exponents for the discrete dynamical system:

$$x_n = T_n x_{n-1}, \qquad n = 1, 2, \ldots \tag{18}$$

where T_n is the transfer matrix of the nth element, which is a random matrix when the structure is randomly disordered.

To evaluate the largest Lyapunov exponent λ_1, a unit initial vector v_0 is chosen, i.e., $\|v_0\| = 1$. The solution of Eq. (18) is evaluated iteratively or element by element. At the kth element,

$$u_k = T_k v_{k-1} \tag{19}$$

The new vector u_k is then normalized,

$$v_k = \frac{u_k}{\|u_k\|}, \quad \text{or} \quad u_k = \|u_k\| v_k \tag{20}$$

Therefore, at the Nth element, the magnitude of v_0 evolves to

$$\begin{aligned} \|T_N T_{N-1} \ldots T_1 v_0\| &= \|T_N T_{N-1} \ldots T_2 u_1\| \\ &= \|T_N T_{N-1} \ldots T_2 v_1\| \cdot \|u_1\| \\ &= \|T_N T_{N-1} \ldots T_3 u_2\| \cdot \|u_1\| \\ &= \|T_N T_{N-1} \ldots T_3 v_2\| \cdot \|u_2\| \cdot \|u_1\| \\ &\ldots \quad \ldots \\ &= \|u_N\| \cdot \|u_{N-1}\| \ldots \|u_1\| \end{aligned} \tag{21}$$

Since it is known that the length of a unit initial vector evolves on the average as $e^{\lambda_1 N}$, the largest Lyapunov exponent is given by

$$\lambda_1 = \lim_{N \to \infty} \frac{1}{N} \ln (\|u_N\| \cdot \|u_{N-1}\| \ldots \|u_1\|)$$

or

$$\lambda_1 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \ln (\|u_k\|) \tag{22}$$

In general, the m-dimensional volume defined by the first m principal axes evolves in the system (18) on the average as $\exp(\lambda_1 + \lambda_2 + \cdots + \lambda_m)N$. Thus $\lambda_1 + \lambda_2 + \cdots + \lambda_m$ can be found by following the evolution of any m linearly independent vectors. However, almost all vectors tend to point toward the local direction of most rapid growth as they evolve, i.e., at the rate of the largest Lyapunov exponent. This problem can be overcome by applying the Gram-Schmidt orthonormalization procedure. To evaluate the mth Lyapunov exponent, m orthogonal unit vectors $v_0^{(1)}, v_0^{(2)}, \ldots, v_0^{(m)}$ are chosen as the initial conditions. The solution of Eq. (18) is evaluated iteratively for each initial vector. At the kth element,

$$u_k^{(i)} = T_k v_{k-1}^{(i)}, \qquad i = 1, 2, \ldots, m \tag{23}$$

The vectors $u_k^{(i)}, i = 1, 2, \ldots, m$ are usually not orthogonal. The Gram-Schmidt orthonormalization procedure is now applied:

$$\hat{u}_k^{(1)} = u_k^{(1)}, \qquad v_k^{(1)} = \frac{\hat{u}_k^{(1)}}{\|\hat{u}_k^{(1)}\|}$$

$$\hat{u}_k^{(2)} = u_k^{(2)} - (u_k^{(2)}, v_k^{(1)})\, v_k^{(1)}, \qquad v_k^{(2)} = \frac{\hat{u}_k^{(2)}}{\|\hat{u}_k^{(2)}\|}$$

$$\cdots \quad \cdots \tag{24}$$

$$\hat{u}_k^{(m)} = u_k^{(m)} - (u_k^{(m)}, v_k^{(m-1)})\, v_k^{(m-1)} - (u_k^{(m)}, v_k^{(m-2)})\, v_k^{(m-2)}$$

$$- \cdots - (u_k^{(m)}, v_k^{(1)})\, v_k^{(1)}$$

$$v_k^{(m)} = \frac{\hat{u}_k^{(m)}}{\|\hat{u}_k^{(m)}\|}$$

where $(\cdot, \cdot)$ denotes the vector dot product.

After the transfer matrix T_k or after the kth element, the volume of an m-dimensional unit hypersphere becomes

$$\|\hat{u}_k^{(1)}\| \cdot \|\hat{u}_k^{(2)}\| \cdots \|\hat{u}_k^{(m)}\| = \prod_{i=1}^{m} \|\hat{u}_k^{(i)}\|$$

Hence, after a product of transfer matrices $T_N T_{N-1} \ldots T_1 = \Pi_{k=1}^{N} T_k$ or after N elements, the volume of an m-dimensional unit hypersphere is

$$\prod_{k=1}^{N} \left[\prod_{i=1}^{m} \|\hat{u}_k^{(i)}\| \right]$$

From the definition of the Lyapunov exponents, one has

$$\exp(\lambda_1 + \lambda_2 + \cdots + \lambda_m)N = \prod_{k=1}^{N} \left[\prod_{i=1}^{m} \|\hat{u}_k^{(i)}\| \right], \qquad \text{for } N \to \infty$$

since, as mentioned before, the m-dimensional volume defined by the first m principal axes evolves on the average as $\exp(\lambda_1 + \lambda_2 + \cdots + \lambda_m)N$. Therefore

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{m} \left[\sum_{k=1}^{N} \ln \|\hat{u}_k^{(i)}\| \right]$$

$$= \lambda_1 + \lambda_2 + \cdots + \lambda_{m-1} + \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{m} \ln \|\hat{u}_k^{(m)}\|$$

so that

$$\lambda_m = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \ln \|\hat{u}_k^{(m)}\| \tag{25}$$

By following the evolution of a $2d$-dimensional hypersphere, each of the $2d$ Lyapunov exponents can be calculated.

In the implementation of this algorithm, certain properties of each element of the structure, such as span length L, bending stiffness EI, etc., are assumed to be randomly disordered and are therefore random variables (see Sec. IV). For each element, random numbers are generated for these quantities, and the corresponding random transfer matrix is evaluated. Since the Lyapunov exponents are quantities characterizing the average exponential rates at which the amplitudes of the waves propagating in the structures decay, Eq. (25) is valid only for large N. It is important that a large value of N be used, since the magnitudes of the smallest positive Lyapunov exponent are usually very small. For the numerical simulation in the following section, N is taken as 10^6.

IV. Wave Localization of Large Beamlike Lattice Trusses

As an example of multiwave structures, the large beamlike lattice truss (Fig. 3) used in large-area space structures such as a space solar power station is considered. The original lattice structure is replaced by an equivalent continuous Timoshenko beam in which the effects of shear deflection and rotary inertia are also taken into consideration. In terms of the dimensions and the properties of the original lattice truss, the properties of the equivalent Timoshenko beam are determined as follows[17]:

Mass per unit length μ:

$$\mu = \rho A = 4\left(\rho_L A_L + \frac{b}{L}\rho_b A_b + \frac{2d}{L}\rho_d A_d\right), \qquad d = \sqrt{b^2 + L^2}$$

Flexural rigidity EI:

$$EI = b^2\left(E_L A_L + \frac{L^3}{\mu_1 d^3} E_d A_d\right), \qquad \mu_1 = 1 + 2\frac{b^3}{d^3}\frac{E_d A_d}{E_b A_b}$$

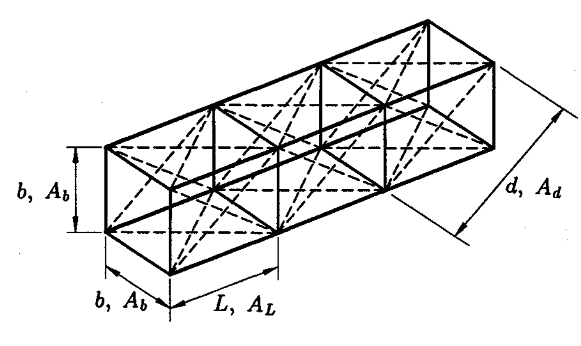
Shear stiffness GA_s:

$$GA_s = \frac{4b^2 L}{d^3} E_d A_d$$

Radius of gyration r:

$$r^2 = \frac{\rho I}{\rho A}, \qquad \rho I = b^2\left(\rho_L A_L + \frac{2b}{3L}\rho_b A_b + \frac{4d}{3L}\rho_d A_d\right)$$

It is assumed that the length L of the longitudinal bars is a uniformly distributed random number with mean L_0 and standard deviation σ_L; the length of the diagonal bars is assumed to vary accordingly and satisfy $d = \sqrt{b^2 + L^2}$. The large beamlike lattice truss may be considered as a randomly disordered periodic structure with each element being an equivalent Timoshenko beam of the corresponding lattice truss element. The transfer matrix of a Timoshenko beam element is derived in the Appendix.



	Cross-Sectional Area	Length	Material Mass Density	Young's Modulus
Longitudinal Bars	A_L	L	ρ_L	E_L
Diagonal Bars	A_d	d	ρ_d	E_d
Battens	A_b	b	ρ_b	E_b

Fig. 3 Beamlike lattice truss.

For the periodic structure, i.e., when $\sigma_L = 0$, the eigenvalues are solutions of the determinantal equation $\det(m\boldsymbol{I}-\boldsymbol{T}) = 0$, or, after expansion of the determinant,

$$m^4 - \mathrm{tr}(\boldsymbol{T})\, m^3 + \alpha m^2 + \beta m + \det(\boldsymbol{T}) = 0 \tag{26}$$

where

$$\alpha = t_{11}t_{22} + t_{11}t_{33} + t_{11}t_{44} + t_{22}t_{33} + t_{22}t_{44} + t_{33}t_{44}$$
$$- (t_{12}t_{21} + t_{13}t_{31} + t_{14}t_{41} + t_{23}t_{32} + t_{24}t_{42} + t_{34}t_{43})$$

On the other hand, it is known that the eigenvalues of the transfer matrix $\boldsymbol{T}$ are reciprocal pairs, i.e., the eigenvalues are of the form m_1, $1/m_1$, m_2, $1/m_2$. Hence,

$$\det(m\boldsymbol{I} - \boldsymbol{T}) = (m - m_1)\left(m - \frac{1}{m_1}\right)(m - m_2)\left(m - \frac{1}{m_2}\right)$$
$$= m^4 - \left(m_1 + m_2 + \frac{1}{m_1} + \frac{1}{m_2}\right)m^3$$
$$+ \left(2 + m_1 m_2 + \frac{m_1}{m_2} + \frac{m_2}{m_1} + \frac{1}{m_1 m_2}\right)m^2$$
$$- \left(m_1 + m_2 + \frac{1}{m_1} + \frac{1}{m_2}\right)m + 1$$

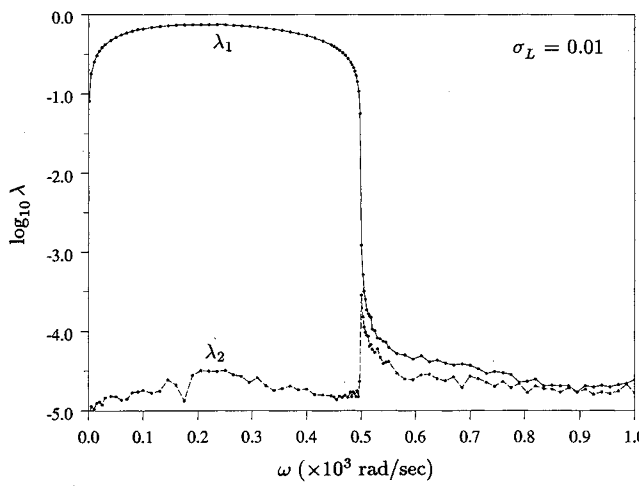


Fig. 4 Lyapunov exponents for Timoshenko beam.

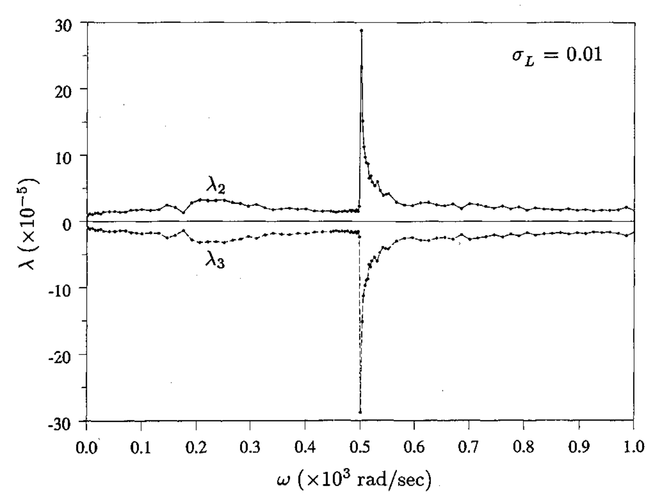


Fig. 5 Lyapunov exponents for Timoshenko beam.

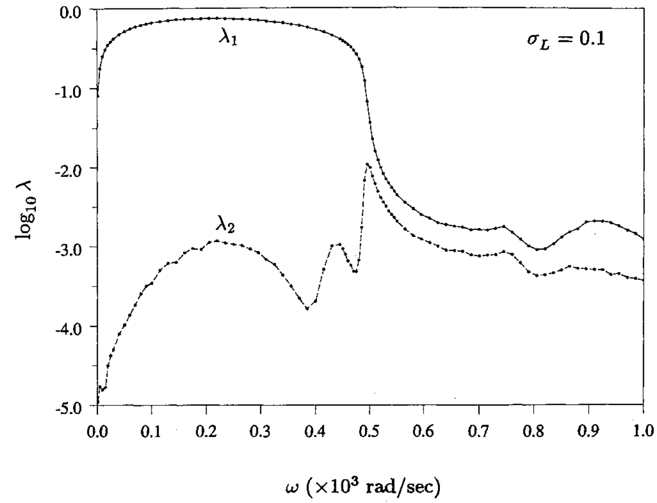


Fig. 6 Lyapunov exponents for Timoshenko beam.

which leads to $\beta = -\mathrm{tr}(\boldsymbol{T})$, and $\det(\boldsymbol{T}) = 1$. The eigenvalue Eq. (26) becomes

$$m^4 - \mathrm{tr}(\boldsymbol{T})\, m^3 + \alpha m^2 - \mathrm{tr}(\boldsymbol{T})\, m + 1 = 0 \tag{27}$$

Dividing both sides of the eigenequation by m^2 gives

$$M^2 - \mathrm{tr}(\boldsymbol{T})\, M + (\alpha - 2) = 0 \tag{28}$$

where $M = m + 1/m$.

The solutions of Eq. (28) are given by

$$M_1 = \frac{\mathrm{tr}(\boldsymbol{T}) + \sqrt{[\mathrm{tr}(\boldsymbol{T})]^2 - 4(\alpha - 2)}}{2}$$
$$M_2 = \frac{\mathrm{tr}(\boldsymbol{T}) - \sqrt{[\mathrm{tr}(\boldsymbol{T})]^2 - 4(\alpha - 2)}}{2} \tag{29}$$

Case 1: If $[\mathrm{tr}(T)]^2 \geq 4(\alpha-2)$, M_1 and M_2 Are Real

In terms of M_1 and M_2, the eigenvalues of Eq. (27) are of the form:

$$m_1 = \frac{M_1}{2} + \sqrt{\left(\frac{M_1}{2}\right)^2 - 1}, \quad \frac{1}{m_1} = \frac{M_1}{2} - \sqrt{\left(\frac{M_1}{2}\right)^2 - 1}$$
$$m_2 = \frac{M_2}{2} + \sqrt{\left(\frac{M_2}{2}\right)^2 - 1}, \quad \frac{1}{m_2} = \frac{M_2}{2} - \sqrt{\left(\frac{M_2}{2}\right)^2 - 1} \tag{30}$$

1. $|M_1| \geq 2, |M_2| \geq 2$

The eigenvalues of the transfer matrix are real and of the form $e^{\pm a}$ or $e^{\pm a + i\pi}$ ($a \in \boldsymbol{R}$); the corresponding frequencies are in the stopband, and the wave amplitudes of all waves after traversing n elements are attenuated by the factor $e^{\pm an}$, in which the real exponent a implies nontraveling or attenuating waves.

2. $|M_1| < 2, |M_2| < 2$

The eigenvalues of the transfer matrix are complex and of the form $e^{\pm ik}$; the corresponding frequencies are in the passband, and all waves travel in the form of $e^{\pm ikn}$, where k is the real wave number, the positive sign indicating left-traveling waves and the negative sign right-traveling waves.

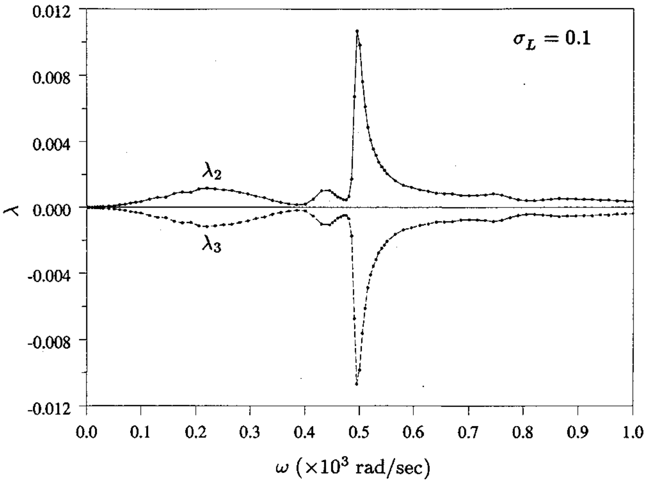


Fig. 7 Lyapunov exponents for Timoshenko beam.

3. $|M_1| \geq 2, |M_2| < 2$

The eigenvalues m_1 and $1/m_1$ are real and of the form $e^{\pm a}$ or $e^{\pm a + i\pi}$ ($a \in \boldsymbol{R}$); the corresponding frequencies are in the stopband, and the wave is a nontraveling wave. The eigenvalues m_2 and $1/m_2$ are complex and of the form $e^{\pm ik}$; the corresponding frequencies are in the passband, and the wave is a traveling wave.

4. $|M_1| < 2, |M_2| \geq 2$

The eigenvalues m_1 and $1/m_1$ are complex and of the form $e^{\pm ik}$; the corresponding frequencies are in the passband, and the wave is a traveling wave. The eigenvalues m_2 and $1/m_2$ are real and of the form $e^{\pm a}$ or $e^{\pm a + i\pi}$ ($a \in \boldsymbol{R}$); the corresponding frequencies are in the stopband, and the wave is a nontraveling wave.

Case 2: If $[\mathrm{tr}(T)]^2 < 4(\alpha - 2)$, M_1 and M_2 Are Complex

The eigenvalues of the transfer matrix are complex and of the form $e^{\pm ik}$; the corresponding frequencies are in the passband, and all waves are traveling waves.

As a numerical example, the following values are chosen: $L_0 = 7.5$ m, $b = 5$ m, $A_L = 80 \times 10^{-6}$ m^2, $A_b = 60 \times 10^{-6}$ m^2, $A_d = 40 \times 10^{-6}$ m^2, $E_L = E_b = E_d = 71.7 \times 10^9$ N/m^2, and $\rho_L = \rho_b = \rho_d = 2768$ kg/m^3. It is found that, for all values of ω, $[\mathrm{tr}(T)]^2 > 4(\alpha - 2)$ in Eqs. (29). When $\omega < 498.54$ rad/s, $|M_1| > 2$ and $|M_2| < 2$, which leads to case 1.3, i.e., one wave is traveling and the other is nontraveling. When $\omega > 498.54$ rad/s, $|M_1| < 2$ and $|M_2| < 2$, which leads to case 1.2, i.e., both waves are traveling.

Using the algorithm described in Sec. III, all four Lyapunov exponents of the randomly disordered Timoshenko beam are determined for $0 < \omega \leq 1000$ rad/s, $N = 10^6$, $\sigma_L = 0.01$, and $\sigma_L = 0.1$, respectively, with the transfer matrix given by Eqs. (A14) or (A20). From Sec. II, it is known that the Lyaponov exponents of the randomly disordered Timoshenko beam are of the form λ_1, λ_2, $\lambda_3 = -\lambda_2$, and $\lambda_4 = -\lambda_1$, ($\lambda_1 > \lambda_2$). This property may be used to check the correctness of the algorithm. The numerical results obtained for the Lyapunov exponents indeed show that this condition is satisfied very well.

The numerical results are plotted in Fig. 4–7. For all frequencies, all of the waves are localized due to the randomness in the length of the horizontal bars. It is seen that the larger the disorder parameter σ_L, the larger the localization factors. The smallest positive Lyapunov exponent λ_2 is of particular interest, since it gives the smallest localization factor, which corresponds to the wave that carries energy along the structure farther than the wave corresponding to λ_1. The magnitudes of the localization factors reflect the degrees at which the waves are localized.

V. Conclusion

The localization of multiwave structures has been studied in this paper. The structure was modeled by the transfer matrices of its elements. For a perfectly periodic multiwave structure, in a given frequency band, some of the waves may be traveling waves and the others nontraveling waves. When the structure is randomly disordered due to imperfection in geometry or material properties, all of the waves are attenuated. The exponential rates of decay of the amplitudes of the waves propagating in the structure are characterized by the Lyapunov exponents. An algorithm was introduced to determine all of the Lyapunov exponents. The smallest positive Lyapunov exponent is of particular interest; it is related to the smallest localization factor or the largest localization length and corresponds to the wave that has potentially the least amount of decay or that carries energy along the multiwave structure farther than any other waves. As an application, the localization of a large beamlike lattice truss used in large-area space structures was studied. The skeleton lattice truss was modeled as an equivalent continuous Timoshenko beam.

Appendix: Transfer Matrix for the Flexural Vibration of a Timoshenko Beam

The equation of motion for the flexural deflection $w(x, t)$ of a Timoshenko beam in which the effects of shear deformation and rotary inertia are taken into consideration is given by (see, e.g., Ref. 14)

$$\frac{EI}{\mu}\frac{\partial^4 w}{\partial x^4} - \left(\frac{EI}{GA_s} + r^2\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^2 w}{\partial t^2} + \frac{\mu r^2}{GA_s}\frac{\partial^4 w}{\partial t^4} = 0, \qquad 0 \leq x \leq L \tag{A1}$$

where L is the length of the beam, EI is the flexural rigidity, $GA_s = GA\kappa$ is the shear stiffness, κ is the Timoshenko shear coefficient depending on the shape of the cross-section, A is the cross-sectional area, μ is the mass per unit length of the beam, and $r = \sqrt{I/A}$ is the radius of gyration. Letting $w(x, t) = W(x)e^{i\omega t}$, Eq. (A1) becomes

$$\frac{d^4 W}{dx^4} + \frac{\mu\omega^2}{EI}\left(\frac{EI}{GA_s} + r^2\right)\frac{d^2 W}{dx^2} - \frac{\mu\omega^2}{EI}\left(1 - \frac{\mu r^2 \omega^2}{GA_s}\right)W = 0 \tag{A2}$$

Employing the notations

$$\sigma = \frac{\mu\omega^2 L^2}{GA_s}, \qquad \tau = \frac{\mu r^2 \omega^2 L^2}{EI}, \qquad \beta^4 = \frac{\mu\omega^2 L^4}{EI}, \qquad \xi = \frac{x}{L}$$

Eq. (A2) takes a simpler form:

$$\frac{d^4 W}{d\xi^4} + (\sigma + \tau)\frac{d^2 W}{d\xi^2} - (\beta^4 - \sigma\tau)W = 0, \qquad 0 \leq \xi \leq 1 \tag{A3}$$

Let $W(\xi) = Ce^{\lambda\xi}$, where λ is the characteristic value satisfying the characteristic equation

$$\lambda^4 + (\sigma + \tau)\lambda^2 - (\beta^4 - \sigma\tau) = 0 \tag{A4}$$

The solutions of Eq. (A4) are given by

$$\lambda^2 = -1/2(\sigma + \tau) \pm \sqrt{\beta^4 + 1/4(\sigma - \tau)^2} \tag{A5}$$

Case 1: $-1/2(\sigma + \tau) + \sqrt{\beta^4 + 1/4(\sigma - \tau)^2} < 0$

The roots of Eq. (A4) are $\pm i\lambda_1$, $\pm i\lambda_2$, where

$$\lambda_1 = \sqrt{1/2(\sigma + \tau) - \sqrt{\beta^4 + 1/4(\sigma - \tau)^2}}$$
$$\lambda_2 = \sqrt{1/2(\sigma + \tau) + \sqrt{\beta^4 + 1/4(\sigma - \tau)^2}} \tag{A6}$$

Hence, the deflection $W(\xi)$ is of the form

$$W(\xi) = C_1\cos\lambda_1\xi + C_2\sin\lambda_1\xi + C_3\cos\lambda_2\xi + C_4\sin\lambda_2\xi$$

Since the shear force $V(\xi)$ is related to the deflection $W(\xi)$ by

$$\frac{dV}{d\xi} = -\mu\omega^2 LW(\xi)$$

one may assume that

$$V(\xi) = A_1\cos\lambda_1\xi + A_2\sin\lambda_1\xi + A_3\cos\lambda_2\xi + A_4\sin\lambda_2\xi \quad \text{(A7)}$$

The deflection $W(\xi)$ is then given by

$$W(\xi) = -\frac{L^3}{\beta^4 EI}(-A_1\lambda_1\sin\lambda_1\xi + A_2\lambda_1\cos\lambda_1\xi - A_3\lambda_2\sin\lambda_2\xi + A_4\lambda_2\cos\lambda_2\xi) \quad \text{(A8)}$$

in which the relationship $\omega^2 = \beta^4 EI/(\mu L^4)$ has been utilized.

The angle of rotation $\psi(\xi)$ is related to $V(\xi)$ and $W(\xi)$ by

$$\psi(\xi) = \frac{\sigma L^2}{\beta^4 EI}V(\xi) - \frac{1}{L}\frac{dW(\xi)}{d\xi}$$

which leads to

$$\psi(\xi) = \frac{L^2(\sigma-\lambda_1^2)}{\beta^4 EI}(A_1\cos\lambda_1\xi + A_2\sin\lambda_1\xi) + \frac{L^2(\sigma-\lambda_2^2)}{\beta^4 EI}(A_3\cos\lambda_2\xi + A_4\sin\lambda_2\xi) \quad \text{(A9)}$$

The bending moment $M(\xi)$ is related to $\psi(\xi)$ by

$$M(\xi) = \frac{EI}{L}\frac{d\psi(\xi)}{d\xi}$$

which results in

$$M(\xi) = \frac{L\lambda_1(\sigma-\lambda_1^2)}{\beta^4}(-A_1\sin\lambda_1\xi + A_2\cos\lambda_1\xi) + \frac{L\lambda_2(\sigma-\lambda_2^2)}{\beta^4}(-A_3\sin\lambda_2\xi + A_4\cos\lambda_2\xi) \quad \text{(A10)}$$

Eqs. (A7–A10) can be written in the matrix form

$$z(\xi) = B(\xi)A \quad \text{(A11)}$$

where

$$z(\xi) = \{-W(\xi), \psi(\xi), M(\xi), V(\xi)\}^T$$

$$A = \{A_1, A_2, A_3, A_4\}^T$$

$$B(\xi) = \begin{bmatrix} -\frac{L^3\lambda_1}{\beta^4 EI}\sin\lambda_1\xi & \frac{L^3\lambda_1}{\beta^4 EI}\cos\lambda_1\xi & -\frac{L^3\lambda_2}{\beta^4 EI}\sin\lambda_2\xi & \frac{L^3\lambda_2}{\beta^4 EI}\cos\lambda_2\xi \\ \frac{L^2(\sigma-\lambda_1^2)}{\beta^4 EI}\cos\lambda_1\xi & \frac{L^2(\sigma-\lambda_1^2)}{\beta^4 EI}\sin\lambda_1\xi & \frac{L^2(\sigma-\lambda_2^2)}{\beta^4 EI}\cos\lambda_2\xi & \frac{L^2(\sigma-\lambda_2^2)}{\beta^4 EI}\sin\lambda_2\xi \\ -\frac{L\lambda_1(\sigma-\lambda_1^2)}{\beta^4}\sin\lambda_1\xi & \frac{L\lambda_1(\sigma-\lambda_1^2)}{\beta^4}\cos\lambda_1\xi & -\frac{L\lambda_2(\sigma-\lambda_2^2)}{\beta^4}\sin\lambda_2\xi & \frac{L\lambda_2(\sigma-\lambda_2^2)}{\beta^4}\cos\lambda_2\xi \\ \cos\lambda_1\xi & \sin\lambda_1\xi & \cos\lambda_2\xi & \sin\lambda_2\xi \end{bmatrix}$$

At the point $\xi = 0$, $z_{i-1} = z(\xi)|_{\xi=0} = B(0)A$, which gives

$$A = B^{-1}(0)z_{i-1} \quad \text{(A12)}$$

At the point $\xi = 1$, $z_i = z(\xi)|_{\xi=1} = B(1)A$. Substituting in Eq. (A12) results in

$$z_i = B(1)B^{-1}(0)z_i = T_{i-1}z_{i-1} \quad \text{(A13)}$$

and $T_i = B(1)B^{-1}(0)$ is the transfer matrix obtained as

$$T_i = \begin{bmatrix} c_0+\sigma c_2 & Lc_1 & -\frac{L^2}{EI}c_2 & -\frac{L^3}{\beta^4 EI}[\sigma c_1 - (\sigma\tau-\beta^4)c_3] \\ -\frac{\beta^4}{L}c_3 & c_0+\tau c_2 & \frac{L}{EI}(c_1-\sigma c_3) & -\frac{L^2}{EI}c_2 \\ -\frac{\beta^4 EI}{L^2}c_2 & \frac{EI}{L}(\sigma c_1 - c_4) & c_0+\tau c_2 & Lc_1 \\ \frac{\beta^4 EI}{L^3}(c_1-\tau c_3) & -\frac{\beta^4 EI}{L^2}c_2 & -\frac{\beta^4}{L^2}c_3 & c_0+\sigma c_2 \end{bmatrix} \quad \text{(A14)}$$

where

$$c_0 = \Lambda(\lambda_1^2\cos\lambda_2 - \lambda_2^2\cos\lambda_1), \qquad c_1 = \Lambda(\lambda_1\sin\lambda_1 - \lambda_2\sin\lambda_2)$$

$$c_2 = \Lambda(\cos\lambda_1 - \cos\lambda_2), \qquad c_3 = \Lambda\left(\frac{\sin\lambda_1}{\lambda_1} - \frac{\sin\lambda_2}{\lambda_2}\right)$$

$$c_4 = \Lambda(\lambda_1^3\sin\lambda_1 - \lambda_2^3\sin\lambda_2)$$

$$\Lambda = \frac{1}{\lambda_1^2 - \lambda_2^2} = -\frac{1}{2\sqrt{\beta^4 + 1/4\,(\sigma - \tau)^2}}$$

Case 2: $-1/2\,(\sigma + \tau) + \sqrt{\beta^4 + 1/4\,(\sigma - \tau)^2} > 0$.

The roots of Eq. (A4) are $\pm\lambda_1$ and $\pm i\lambda_2$, where

$$\lambda_1 = \sqrt{-1/2\,(\sigma + \tau) + \sqrt{\beta^4 + 1/4\,(\sigma - \tau)^2}}$$

$$\lambda_2 = \sqrt{1/2\,(\sigma + \tau) + \sqrt{\beta^4 + 1/4\,(\sigma - \tau)^2}} \qquad \text{(A15)}$$

Following the same procedure as in case 1, it can be shown that

$$V(\xi) = A_1 \cosh \lambda_1\xi + A_2 \sinh \lambda_1\xi + A_3 \cos \lambda_2\xi + A_4 \sin \lambda_2\xi \qquad \text{(A16)}$$

$$W(\xi) = -\frac{L^3}{\beta^4 EI}(A_1\lambda_1 \sinh \lambda_1\xi + A_2\lambda_1 \cosh \lambda_1\xi - A_3\lambda_2 \sin \lambda_2\xi + A_4\lambda_2 \cos \lambda_2\xi) \qquad \text{(A17)}$$

$$\psi(\xi) = \frac{L^2(\sigma + \lambda_1^2)}{\beta^4 EI}(A_1 \cosh \lambda_1\xi + A_2 \sinh \lambda_1\xi) + \frac{L^2(\sigma - \lambda_2^2)}{\beta^4 EI}(A_3 \cos \lambda_2\xi + A_4 \sin \lambda_2\xi) \qquad \text{(A18)}$$

$$M(\xi) = \frac{L\lambda_1(\sigma + \lambda_1^2)}{\beta^4}(A_1 \sinh \lambda_1\xi + A_2 \cosh \lambda_1\xi) + \frac{L\lambda_2(\sigma - \lambda_2^2)}{\beta^4}(-A_3 \sin \lambda_2\xi + A_4 \cos \lambda_2\xi) \qquad \text{(A19)}$$

In the matrix form of Eq. (A11), the matrix $\boldsymbol{B}(\xi)$ is given by

$$\boldsymbol{B}(\xi) = \begin{bmatrix} \frac{L^3\lambda_1}{\beta^4 EI}\sinh \lambda_1\xi & \frac{L^3\lambda_1}{\beta^4 EI}\cosh \lambda_1\xi & -\frac{L^3\lambda_2}{\beta^4 EI}\sin \lambda_2\xi & \frac{L^3\lambda_2}{\beta^4 EI}\cos \lambda_2\xi \\ \frac{L^2(\sigma + \lambda_1^2)}{\beta^4 EI}\cosh \lambda_1\xi & \frac{L^2(\sigma + \lambda_1^2)}{\beta^4 EI}\sinh \lambda_1\xi & \frac{L^2(\sigma - \lambda_2^2)}{\beta^4 EI}\cos \lambda_2\xi & \frac{L^2(\sigma - \lambda_2^2)}{\beta^4 EI}\sin \lambda_2\xi \\ \frac{L\lambda_1(\sigma + \lambda_1^2)}{\beta^4}\sinh \lambda_1\xi & \frac{L\lambda_1(\sigma + \lambda_1^2)}{\beta^4}\cosh \lambda_1\xi & -\frac{L\lambda_2(\sigma - \lambda_2^2)}{\beta^4}\sin \lambda_2\xi & \frac{L\lambda_2(\sigma - \lambda_2^2)}{\beta^4}\cos \lambda_2\xi \\ \cosh \lambda_1\xi & \sinh \lambda_1\xi & \cos \lambda_2\xi & \sin \lambda_2\xi \end{bmatrix}$$

The transfer matrix $\boldsymbol{T}_i = \boldsymbol{B}(1)\boldsymbol{B}^{-1}(0)$ may be obtained as

$$\boldsymbol{T}_i = \begin{bmatrix} c_0 - \sigma c_2 & Lc_1 & \frac{L^2}{EI}c_2 & -\frac{L^3}{\beta^4 EI}[\sigma c_1 + (\sigma\tau - \beta^4)c_3] \\ \frac{\beta^4}{L}c_3 & c_0 - \tau c_2 & \frac{L}{EI}(c_1 + \sigma c_3) & \frac{L^2}{EI}c_2 \\ \frac{\beta^4 EI}{L^2}c_2 & \frac{EI}{L}(\sigma c_1 + c_4) & c_0 - \tau c_2 & Lc_1 \\ \frac{\beta^4 EI}{L^3}(c_1 + \tau c_3) & \frac{\beta^4 EI}{L^2}c_2 & \frac{\beta^4}{L^2}c_3 & c_0 - \sigma c_2 \end{bmatrix} \qquad \text{(A20)}$$

where

$$c_0 = \Lambda(\lambda_1^2 \cosh \lambda_2 + \lambda_2^2 \cos \lambda_1), \qquad c_1 = \Lambda(\lambda_1 \sinh \lambda_1 + \lambda_2 \sin \lambda_2)$$

$$c_2 = \Lambda(\cosh \lambda_1 - \cos \lambda_2), \qquad c_3 = \Lambda\left(\frac{\sinh \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2}\right)$$

$$c_4 = \Lambda(\lambda_1^3 \sinh \lambda_1 - \lambda_2^3 \sin \lambda_2)$$

$$\Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2} = \frac{1}{2\sqrt{\beta^4 + 1/4\,(\sigma - \tau)^2}}$$

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References

[1]Anderson, P. W., "Absence of Diffusion in Certain Random Lattices," *Physical Review: A Journal of Experimental and Theoretical Physics*, Vol. 109, No. 5, 1958, pp. 1492–1505.

[2]Hodges, C. H., "Confinement of Vibration by Structural Irregularity," *Journal of Sound and Vibration*, Vol. 82, No. 3, 1982, pp. 411–424.

[3]Pierre, C., and Dowell, E. H., "Localization of Vibrations by Structural Irregularity," *Journal of Sound and Vibration*, Vol. 114, No. 3, 1987, pp. 549–564.

[4]Pierre, C., "Weak and Strong Vibration Localization in Disordered Structures: A Statistical Investigation," *Journal of Sound and Vibration*, Vol. 139, No. 1, 1990, pp. 111–132.

[5]Cha, P. D., and Pierre, C., "Vibration Localization by Disorder in Assemblies of Monocoupled, Multimode Component Systems," *ASME Journal of Applied Mechanics*, Vol. 58, No. 4, 1991, pp. 1072–1081.

[6]Cai, G. Q., and Lin, Y. K., "Localization of Wave Propagation in Disordered Periodic Structures," *AIAA Journal*, Vol. 29, No. 3, 1991, pp. 450–456.

[7]Kissel, G. J., "Localization in Disordered Periodic Structures," Ph.D. Thesis, Dept. of Aeronautics and Astronautics, Massachusetts Inst. of Technology, Cambridge, MA, 1988.

[8]Ariaratnam, S. T., and Xie, W.-C., "Lyapunov Exponents in Stochastic Structural Dynamics," *Lyapunov Exponents (Lecture Notes in Mathematics)*, edited by L. Arnold, H. Crauel, and J.-P. Eckmann, Springer-Verlag, Berlin, 1991, pp. 271–291.

[9]Ariaratnam, S. T., and Xie, W.-C., "Wave Localization in Randomly Disordered Nearly Periodic Long Continuous Beams," *Journal of Sound and Vibration* (to be published).

[10]Furstenberg, H., "Noncommuting Random Products," *Transactions of*

the American Mathematical Society, Vol. 108, No. 3, 1963, pp. 377–428.

[11]Kissel, G. J., "Localization Factor for Multichannel Disordered Systems," *Physical Review A*, Vol. 44, No. 2, 1991, pp. 1008–1014.

[12]Oseledec, Y. I., "A Multiplicative Ergodic Theorem. Lyapunov Characteristic Number for Dynamical Systems," *Transactions of the Moscow Mathematical Society*, Vol. 19, 1968, pp. 197–231 (English translation).

[13]Wolf, A., Swift, J., Swinney, H., and Vastano, A., "Determining Lyapunov Exponents from a Time Series," *Physica D*, Vol. 16, No. 3, 1985, pp. 285–317.

[14]Pestel, E. C., and Leckie, F. A., *Matrix Methods in Elastomechanics*, McGraw-Hill, New York, 1963.

[15]Bougerol, P., and Lacroix, J., *Products of Random Matrices with Applications to Schrödinger Operators*, Birkhäuser, Boston, 1985.

[16]Benettin, G., Galgani, L., Giorgilli, A., and Strelcyn, J.-M., "Lyapunov Characteristic Exponents for Dynamical Systems and for Hamiltonian Systems: A Method for Computing All of Them," *Meccanica*, Vol. 15, No. 1, 1980, pp. 9–30.

[17]Noor, A. K., and Andersen, C. M., "Analysis of Beam-like Lattice Trusses," *Computer Methods in Applied Mechanics and Engineering*, Vol. 20, No. 1, 1979, pp. 53–70.